Written Exam at the Department of Economics winter 2019-20

Financial Econometrics A

Final Exam

February 17, 2020

(3-hour closed book exam)

Answers only in English.

This exam question consists of 6 pages in total

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Please note there is a total of **10** questions that you should provide answers to. That is, **5** questions under *Question A*, and **5** under *Question B*.

Question A:

Consider the model for $x_t \in \mathbb{R}$ (with $t = 1, 2, ..., T$) given by

$$x_t = \mu + \varepsilon_t, \quad \varepsilon_t = \sigma_t z_t,$$

with z_t i.i.d.N$(0, 1)$, $x_0 = 0$ and

$$\sigma_t^2 = \omega + \alpha x_{t-1}^2.$$

The parameters satisfy $\mu \in \mathbb{R}$, $\omega > 0$ and $\alpha \geq 0$.

Question A.1: Derive a condition under which x_t is weakly mixing with $Ex_t^2 < \infty$.

Question A.2: With $\theta = (\mu, \omega, \alpha)'$ the likelihood function is given by

$$L(\theta) = -\frac{1}{2T} \sum_{t=1}^{T} \left(\log \sigma_t^2(\theta) + \frac{(x_t - \mu)^2}{\sigma_t^2(\theta)} \right),$$

with $\sigma_t^2(\theta) = \omega + \alpha x_{t-1}^2$. Show that if $\alpha_0 < 1$, then with $\theta_0 = (\mu_0, \omega_0, \alpha_0)'$ the true parameter value,

$$\sqrt{T} \partial L(\theta_0) / \partial \mu \xrightarrow{D} N(0, \xi) \quad \xi = E\left(\frac{x_t - \mu_0}{\omega_0 + \alpha_0 x_{t-1}^2} \right)^2.$$

Question A.3: Show that if x_t is weakly mixing with $\alpha_0 > 0$, then with $\theta_0 = (\mu_0, \omega_0, \alpha_0)'$ the true parameter value,

$$\sqrt{T} \partial L(\theta_0) / \partial \alpha \xrightarrow{D} N(0, \beta) \quad \beta = \frac{1}{2} E\left(\frac{x_{t-1}^2}{\omega_0 + \alpha_0 x_{t-1}^2} \right)^2.$$

Question A.4: We conclude that if $0 < \alpha_0 < 1$ then asymptotic normality holds for $\hat{\theta}$. Argure that the limiting distribution of the LR statistic for the hypothesis that $\mu = 0$ is χ^2.

Question A.5: Now consider testing the hypothesis that $\alpha = 0$. In this case the asymptotic distribution of the LR statistic is "$\frac{1}{2}\chi^2$". Explain why - and explain how this is related to Questions A.2 and A.3.

Question B:

Suppose that the logarithm of the price of a share of stock is given by

$$p(t) = p(0) + \mu t + \sigma W(t), \quad t \in [0, T], \tag{B.1}$$

where $p(0) \in \mathbb{R}$ is some fixed initial value, $\mu \in \mathbb{R}$ and $\sigma > 0$ are constants, and $W(t)$ is a Brownian motion.

Recall here that the Brownian motion $W(t)$ has the properties

1. $W(0) = 0$.
2. W has independent increments, i.e. if $0 \leq r < s \leq t < u$, then

   $$W(u) - W(t) \text{ and } W(s) - W(r)$$

   are independent.
3. The increments are normally distributed, i.e.

   $$W(t) - W(s) \sim N(0, t - s)$$

   for all $0 \leq s \leq t$.

Suppose that we have observed the price $p(t)$ at $n + 1$ equidistant points

$$0 = t_0 < t_1 < \ldots < t_n = T,$$

with

$$t_i = \frac{i}{n}T, \quad i = 0, ..., n.$$

Based on these points we obtain n log-returns given by

$$r(t_i) = p(t_i) - p(t_{i-1}), \quad i = 1, ..., n.$$

Question B.1: Argue that $r(t_i)$ is normally distributed, i.e. show that

$$r(t_i) \sim N\left(\mu \frac{T}{n}, \sigma^2 \frac{T}{n}\right).$$

Show that

$$\text{cov}(r(t_i), r(t_{i-1})) = 0.$$

Question B.2: We now seek to estimate the model parameters (μ, σ^2) based on maximum likelihood. Given the n log-returns, the log-likelihood function is (up to a constant and a scaling factor)

$$L_n(\mu, \sigma^2) = \sum_{i=1}^{n} \left\{ -\log(\sigma^2 \frac{T}{n}) - \frac{\left[r(t_i) - \mu \frac{T}{n}\right]^2}{\sigma^2 \frac{T}{n}} \right\}.$$

Let $\hat{\mu}$ denote the maximum likelihood estimator of μ.
Show that

$$\hat{\mu} = \frac{1}{T} \sum_{i=1}^{n} r(t_i) = \frac{1}{T} \left[p(T) - p(0)\right].$$

Argue that the sampling frequency of the log-returns over the interval $[0, T]$ does not have any influence on the estimate of μ.

Question B.3: Assume now that $T = 1$, such that we have n observations of the log-returns over the time interval $[0, 1]$, which you may think of as the time interval over one trading day. Then the maximum likelihood estimator for σ^2 is given by

$$\hat{\sigma}^2 = \sum_{i=1}^{n} \left[r(t_i) - \frac{1}{n} \sum_{i=1}^{n} r(t_i) \right]^2.$$

Use that $r(t_i) = \frac{\mu}{n} + \frac{\sigma}{\sqrt{n}} z_i$, with $z_i \sim i.i.d. N(0, 1)$ in order to show that

$$\frac{1}{n} \sum_{i=1}^{n} r(t_i) \xrightarrow{p} 0 \quad \text{as } n \to \infty.$$

Explain briefly how $\hat{\sigma}^2$ is related to the Realized Volatility.

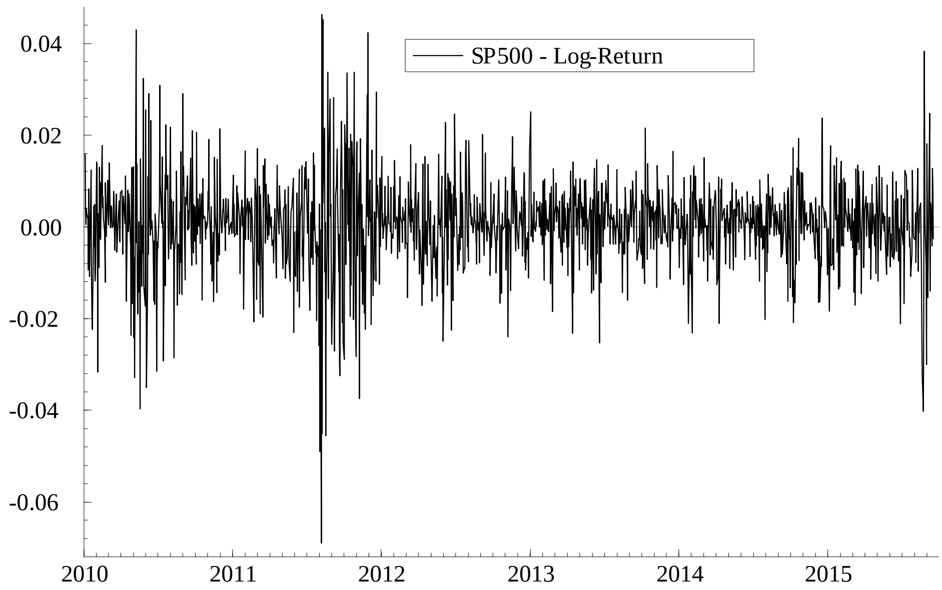
Question B.4: Assume that T is some positive integer ($T \in \mathbb{N}$), and that we have $n = T$ observations of the returns, that is we have a sample $(r(t))_{t=1,\ldots T}$ with $r(t) = p(t) - p(t-1)$. Let

$$\hat{\gamma}_T = \frac{1}{T} \sum_{t=1}^{T} r(t),$$

and argue that as $T \to \infty$,

$$\sqrt{T} \left(\hat{\gamma}_T - \mu\right) \xrightarrow{d} N(0, \sigma^2).$$

Question B.5: The following figure shows the daily log-returns of the S&P 500 index for the period January 4, 2010 to September 17, 2015.



Discuss briefly whether the model in (B.1) is a reasonable model for the daily log returns of the S&P 500 index.